Deep Inelastic Scattering

Vince Bagnulo
Supervisor: J. Martin
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Abstract

The purpose of this text is to describe the process of deep inelastic scattering and how it relates to the structure of the proton. The proton is modelled to consist of quarks and the physical implications of this model for deep inelastic scattering are derived. These physical implications are compared to experimental data and found to be mainly in agreement. Some deviation from experiment is found and it is proposed that this is due to the presence of gluons within the proton.

Chapter 1

Introduction

A scattering experiment in particle physics consists of directing a beam of particles towards a target and determining the probability of deflection into particular directions. Such experiments yield insight into how the two particles interact and are used to learn about the structure of the particles involved. In 1907, Ernest Rutherford performed his famous gold foil experiment in which alpha particles were scattered off of thin gold foil. The alpha particles were found to scatter at unexpectedly large angles. This was the first evidence that the atom must have a massive, positively charged nucleus. In similiar experiments at the Stanford Linear Accelerator Center (SLAC) in 1968, electrons were scattered off proton targets and the results of this experiment yielded evidence for the existence of quarks. The experiment performed at SLAC was an example of a particular type of scattering experiment: deep inelastic scattering. In this type of scattering, high energy electrons are directed towards proton targets with a variety of final state hadrons resulting from the collision. Deep inelastic electron-proton scattering is the subject of this thesis.

In order to discuss deep inelastic scattering some background material must first be introduced. The material that will be described consists of relativistic kinematics, relativistic quantum mechanics, and quantum electrodynamics. Utilizing this material, a model of electron-proton scattering will be presented. This model describes the proton as consisting of spin-1/2, pointlike particles called quarks. Physical implications for deep inelastic scattering will be derived from this model, namely, Bjorken scaling and the Callan-Gross relation. These implications will be shown to be in satisfactory agreement with experiment, thus supporting the hypothesis that the proton consists of quarks. Furthermore, evidence will be presented suggesting that the proton does not consist solely of quarks. An analysis of the momenta of the quarks within the proton will reveal that they do not account for all of the proton momentum. Thus, the proton must possess other constituents. This is evidence for the presence of gluons within the proton.

Chapter 2

Relativistic Kinematics

This chapter will serve to introduce notation and summarize the kinematical laws that will be utilized throughout this text.

1 Notation and Four-Vectors

A Lorentz transformation takes a vector in one coordinate representation into a new coordinate system which is moving at a constant speed v relative to the original coordinate system. In this text, it is assumed that the speed v is in the x-direction and the x axes of the two coordinate systems coincide at $t = 0$; such a Lorentz transformation is called a special Lorentz transformation. These assumptions are made without loss of generality as it is always possible to find a coordinate system such that they are satisfied.

A four-vector is a four-component object that transforms under a Lorentz transformation as follows:

$$a^{\mu\prime} = \Lambda^{\mu}_{\nu} a^{\nu} \tag{2.1}$$

where Greek indices are assumed run from 0 to 3 and repeated indices are summed over. These conventions will be used throughout this text. Λ^{μ}_{ν} is the 4×4 matrix

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

and

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.3}$$

In the above equations, c denotes the speed of light in a vacuum.

The indices of four-vectors can be raised or lowered as follows:

$$a_\mu = g_{\mu\nu} a^\nu, \qquad a^\mu = g^{\mu\nu} a_\nu \tag{2.4}$$

where $g^{\mu\nu}$ is a symmetric 4x4 matrix called a metric. In this text, $g^{\mu\nu}$ is defined as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.5}$$

In three dimensions the dot product is invariant under rotations, similarly, the following quantity is invariant under Lorentz transformations:

$$I = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = g^{\mu\nu} a_\mu a_\nu = a^\mu a_\mu = a \cdot a \tag{2.6}$$

Throughout this text, both four-vectors and three-vectors will be used and so notation distinguishing the two must be introduced. Four-vectors will always be denoted a^μ or, simply, a. The square of their magnitude will be denoted a^2 (a will never be used to denote the magnitude of a four vector). Three-component vectors will be denoted $\mathbf{a}$; their magnitudes will be denoted $|\mathbf{a}|$ or, when squared, $\mathbf{a}^2$.

Objects that have more than one index, such as matrices, can also have their indices raised and lowered and undergo Lorentz transformations. This is performed by applying the relevant operation once for each index. An example utilizing two-index objects is presented below:

$$a^{\mu\nu\prime} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma a^{\lambda\sigma} \qquad a^\mu_\nu = g_{\lambda\nu} a^{\mu\nu} \qquad a_{\mu\nu} = g_{\mu\sigma} g_{\lambda\nu} a^{\sigma\lambda} \tag{2.7}$$

2 Energy and Momentum

In relativistic mechanics, the proper velocity of a particle, denoted η^μ, is defined as follows:

$$\eta^\mu = \gamma \frac{dx^\mu}{dt} \tag{2.8}$$

where x^μ is an equation for the four-position of the particle. Thus, η^μ has the following components:

$$\eta^\mu = \gamma(c, u_x, u_y, u_z) \tag{2.9}$$

where $u_x = dx^1/dt$, $u_y = dx^2/dt$, and $u_z = dx^3/dt$.

The four-momentum of a particle with proper velocity η^μ and mass m is defined:

$$p^\mu = m\eta^\mu \tag{2.10}$$

Consequently, momentum has the following components:

$$p^\mu = (\gamma mc, \gamma mu_x, \gamma mu_y, \gamma mu_z) \tag{2.11}$$

The relativistic equation for the total energy of a particle is:

$$E = \gamma mc^2 \tag{2.12}$$

Thus, the four components of p^μ become:

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) \tag{2.13}$$

Exploiting the invariance of the quantity $p^\mu p_\mu$, it is possible to obtain the following equation:

$$E^2 = \mathbf{p}^2c^2 + (mc^2)^2 \tag{2.14}$$

Eqn. 2.14 will be called the relativistic energy-momentum relation and is the standard equation for determining the energy of a particle in relativistic mechanics. This equation takes on the following form for massless particles:

$$E = |\mathbf{p}|/c \tag{2.15}$$

Eqn. 2.15 will be used frequently in future chapters.

3 Collisions

In special relativity, collisions are classified as inelastic or elastic as follows:

1. elastic - incident particles and resultant particles are the same (for example, Compton scattering: $e^- + \gamma \rightarrow e^- + \gamma$
2. inelastic - resultant particles and incident particles differ (for example, pair production: $\gamma + \gamma \rightarrow e^- + e^+$)

Chapter 3

Scattering Theory

In this chapter, the theory of scattering experiments will be discussed. The differential cross section will be defined and Fermi's Golden Rule, which allows the calculation of differential cross sections, will be introduced. Also, several examples of cross section calculations that will be useful in future chapters will be presented.

1 Differential Cross Section

A differential cross section is a measurement of the probability of two particles interacting. In a classical scattering experiment, if the incident particle passes through the infinitesimal area $d\sigma$ then it will scatter into solid angle $d\Omega$. This is depicted in Figure 3.1. The ratio of these two terms, $\frac{d\sigma}{d\Omega}$, is the differential cross section.

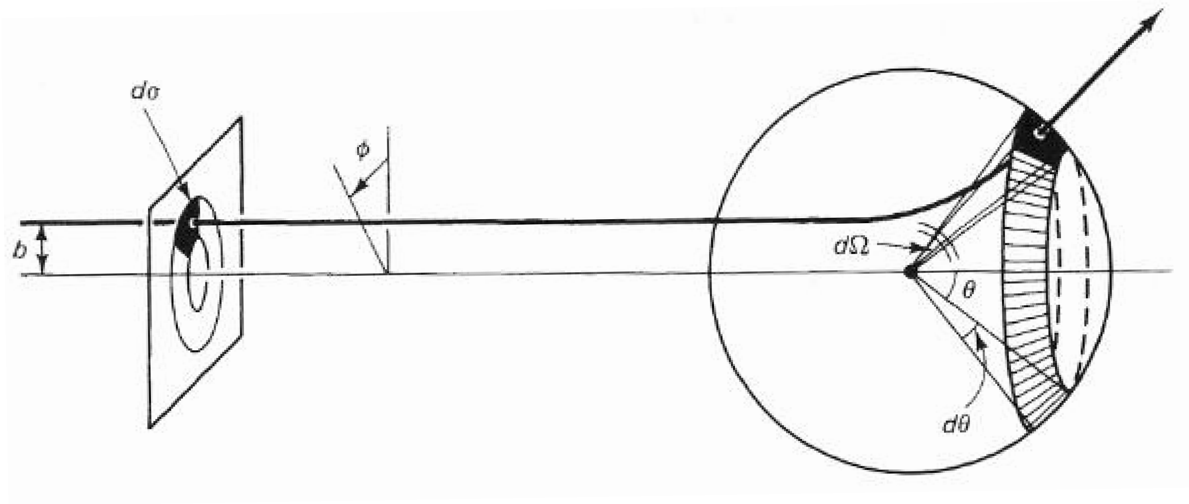

Figure 3.1: Diagram of Differential Cross Section [1]

The differential cross section is useful as it can be determined experimentally. If a beam of incoming particles having uniform luminosity $\mathcal{L}$ is directed towards a target then the number of particles crossing $d\sigma$ per unit time is $dN = \mathcal{L} d\sigma$. Thus:

$$dN = \mathcal{L} d\sigma = \mathcal{L} \frac{d\sigma}{d\Omega} d\Omega \qquad \therefore \qquad \frac{d\sigma}{d\Omega} = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega} \tag{3.1}$$

Both $\mathcal{L}$ and $\frac{dN}{d\Omega}$ are easy to determine experimentally. In quantum mechanics, results of measurements must be expressed statistically and so the differential cross section must be redefined: the differential cross section is the probability of observing a scattered particle to be in a particular quantum state per unit solid angle.

2 Fermi's Golden Rule

Fermi's Golden Rule allows for the calculations of probabilities of transitions; it possesses many forms. In the particular case of scattering, with particles 1 and 2 colliding producing particles $3, 4, \ldots, n$, Fermi's Golden Rule is [1]:

$$\begin{aligned} d\sigma &= |\mathcal{M}|^2 \frac{\hbar^2 S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \left[\left(\frac{c\, d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) \left(\frac{c\, d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \right) \cdots \left(\frac{c\, d^3\mathbf{p}_n}{(2\pi)^3 2E_n} \right) \right] \\ &\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \ldots - p_n) \end{aligned} \tag{3.2}$$

where p_i, $\mathbf{p}_i$, E_i, and m_i are the four-momentum, three-momentum, energy, and mass of particle i, respectively; S is a statistical factor for each group of j identical particles in the final state and $S = 1/j!$; $d^3\mathbf{p} = dp_x\,dp_y\,dp_z$; and $\mathcal{M}$ is the amplitude of the scattering process.

Typically, it is only the angle of p_3 that is of interest and so all of the other momenta and $|\mathbf{p}_3|$ are integrated over. In the next section, Eqn. 3.2 will be applied to particular examples.

3 Examples

In this section two useful examples of collisions will be worked out: two-body scattering in the center of mass frame and in the laboratory frame.

3.1 Two-Body Scattering: Center of Mass Frame

The differential cross section for two-body scattering in the center of mass frame will now be determined. The process is:

$$1 + 2 \rightarrow 3 + 4 \tag{3.3}$$

For two-body scattering, Eqn. 3.2 becomes:

$$d\sigma = |\mathcal{M}|^2 \frac{\hbar^2 S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \left(\frac{c\,d^3\mathbf{p}_3}{(2\pi)^3 2E_3}\right) \left(\frac{c\,d^3\mathbf{p}_4}{(2\pi)^3 2E_4}\right) (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \tag{3.4}$$

but it can be shown that:

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = (E_1 + E_2)|\mathbf{p}_1|/c \tag{3.5}$$

and the delta function can be rewritten:

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta\left(\frac{E_1 + E_2 - E_3 - E_4}{c}\right) \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) = \delta^3(-\mathbf{p}_3 - \mathbf{p}_4) \tag{3.6}$$

The last step can be made because, in the center of mass frame, $\mathbf{p}_1 + \mathbf{p}_2 = 0$. Substituting Eqn. 3.5 and Eqn. 3.6 into Eqn. 3.4 and simplifying yields:

$$d\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2 c}{(E_1+E_2)|\mathbf{p}_1|} \frac{d^3\mathbf{p_3}\, d^3\mathbf{p_4}}{E_3 E_4} \delta\left(\frac{E_1+E_2+E_3+E_4}{c}\right) \delta^3(\mathbf{p}_3+\mathbf{p}_4) \tag{3.7}$$

The energies can be expressed in terms of the three-momenta through the use of Eqn. 2.14. Also, the $\mathbf{p}_4$ integral can be carried out; this has the effect of switching each $\mathbf{p}_4$ with $-\mathbf{p}_3$. Consequently, Eqn. 3.7 becomes:

$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2 c}{(E_1+E_2)|\mathbf{p}_1|} \frac{\delta(E_1/c + E_2/c - \sqrt{m_3^2 c^2 + \mathbf{p}_3^2} - \sqrt{m_4^2 c^2 + \mathbf{p}_3^2}}{\sqrt{m_3^2 c^2 + \mathbf{p}_3^2}\sqrt{m_4^2 c^2 + \mathbf{p}_3^2}} d^3\mathbf{p}_3 \tag{3.8}$$

A change of coordinate system can now be applied: $d^3\mathbf{p}_3 = \mathbf{p}_3^2\, d\mathbf{p_3}\, d\Omega$. As the angular dependence of $|\mathcal{M}|^2$ is unknown, it is not possible to carry out the angular integration and so $d\Omega$ is brought to the other side of the equation. Thus:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\mathbf{p}_1|} \int_0^\infty |\mathcal{M}|^2 \frac{\delta(E_1/c + E_2/c - \sqrt{m_3^2 + \mathbf{p}_3^2} - \sqrt{m_4^2 c^2 + \mathbf{p}_3^2}}{\sqrt{m_3^2 + \mathbf{p}_3^2}\sqrt{m_4^2 c^2 + \mathbf{p}_3^2}} \mathbf{p}_3^2\, d|\mathbf{p}_3| \tag{3.9}$$

To perform the integration, a change of variable is made. Let:

$$E = c\left(\sqrt{m_3^2 c^2 + \mathbf{p}_3^2} - \sqrt{m_4^2 c^2 + \mathbf{p}_3^2}\right) \tag{3.10}$$

Therefore:

$$dE = \frac{E|\mathbf{p}_3|}{\sqrt{m_3^2 c^2 + \mathbf{p}_3^2}\sqrt{m_4^2 c^2 + \mathbf{p}_3^2}} d|\mathbf{p}_3| \tag{3.11}$$

and so:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\mathbf{p}_1|} \int_0^\infty |\mathcal{M}|^2 \delta(E_1/c + E_2/c - E/c) \frac{|\mathbf{p}_3|}{E}\, dE \tag{3.12}$$

Pulling $-1/c$ out of the delta function:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S}{(E_1+E_2)|\mathbf{p}_1|} \int_0^\infty |\mathcal{M}|^2 \delta(E - (E_1+E_2)) \frac{|\mathbf{p}_3|}{E}\, dE \tag{3.13}$$

Carrying out the integration yields the final equation:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \tag{3.14}$$

It should be noted that carrying out the final integral was the same as enforcing conservation of energy. In Eqn. 3.14, $|\mathbf{p}_3|$ is the particular value of the momentum of particle 3 that is consistent with conservation of energy. $|\mathcal{M}|^2$ now depends only on the angle of scattering.

3.2 Two-Body Scattering: Laboratory Frame

The calculation for determining the two-body scattering differential cross section in the laboratory frame is very similiar to the calculation in the center of mass frame. To simplify calculations, it will be assumed that the collision is elastic and the target particle, particle 2, is at rest. Thus, $m_1 = m_3$ and $m_2 = m_4$ and $\mathbf{p}_2 = 0$. The process is:

$$1 + 2 \rightarrow 1 + 2 \tag{3.15}$$

Once again, the starting point is Fermi's Golden Rule for two-body scattering, Eqn. 3.4. However, in this case:

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = m_2 |\mathbf{p}_1| c \tag{3.16}$$

Thus, Eqn. 3.4 becomes:

$$d\sigma = |\mathcal{M}|^2 \frac{\hbar^2 S}{4 m_2 |\mathbf{p}_1| c} \left(\frac{c\, d^3\mathbf{p}_3}{(2\pi)^3 2E_3}\right) \left(\frac{c\, d^3\mathbf{p}_4}{(2\pi)^3 2E_4}\right) (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \tag{3.17}$$

As in the center of mass frame example, it is then possible to rewrite the delta function and integrate over $\mathbf{p}_4$. Subsequently, a change to spherical coordinates can be made to separate the magnitude and angular dependence of $\mathbf{p_3}$. Finally, a change of variables to implement conservation of energy can be made and the final integration carried out. The final result is:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{\mathbf{p}_3^2 S|\mathcal{M}|^2}{m_2 |\mathbf{p}_1| \{|\mathbf{p}_3|(E_1 + m_2 c^2) - |\mathbf{p}_1| E_3 \cos(\theta)\}} \tag{3.18}$$

In the case that the target is very heavy, $m_2 c^2 \gg E_1$, the recoil of particle 1 can be neglected. Thus, $|p_3| \simeq |p_1|$. From this, it can easily be shown that Eqn. 3.18 reduces to

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_2 c}\right)^2 |\mathcal{M}|^2 \tag{3.19}$$

There is another special case that is of interest. If the incident particle is massless, $m_1 = 0$, Eqn. 3.18 can be shown to simplify to:

$$\frac{d\sigma}{d\Omega} = S\left(\frac{\hbar E_3}{8\pi m_2 c E_1}\right)^2 |\mathcal{M}|^2 \tag{3.20}$$

In obtaining Eqn. 3.20 the following equation, which is a consequence of conservation of four-momentum, was used:

$$m_2 E_1 c^2 = E_1 E_3 + m_2 E_3 c^2 - E_1 E_3 \cos\theta \tag{3.21}$$

Chapter 4

Quantum Electrodynamics

1 The Dirac Equation

In non-relativistic quantum mechanics, the time evolution of wave functions is dictated by the Schrodinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \tag{4.1}$$

In relativistic quantum mechanics, the evolution of wave functions is governed by three different equations, based on the spin of the particle being described. Spin-0 particles are governed by the Klein-Gordon Equation, spin-1 particles are governed by the Proca equation, and spin-1/2 particles are governed by the Dirac equation. It is spin-1/2 particles that are primarily of interest in this work and so it is the Dirac equation that will be used. The Dirac equation for free particles is:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0 \tag{4.2}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and the γ^μ are 4×4 matrices called the gamma matrices. There are several sets of equivalent gamma matrices; in this work, the following set will be used:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4.3}$$

where the $\sigma^i (i = 1, 2, 3)$ are the Pauli matrices, 1 is the 2×2 identity matrix and 0 is a 2×2 matrix of zeroes. It is also possible to define a fifth gamma matrix:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.4}$$

Despite the notation, γ^μ, the gamma matrices are not four-vectors. They are simply a collection of four fixed matrices; much like the Pauli matrices, they do not transform under changes of coordinates. However, it is possible to raise and lower the indices of the gamma matrices as is done with four-vectors: $\gamma_\mu = g_{\mu\nu}\gamma^\mu$.

As the γ^μ are 4×4 matrices, ψ must be a four-element column matrix:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{4.5}$$

Although ψ is an object with four components, it is not a four-vector; rather it is what is called a "bi-spinor" or "spinor". A bispinor, ψ, transforms under a Lorentz transformation as follows:

$$\psi' = S\psi \qquad \text{where} \qquad S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} \tag{4.6}$$

and $a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$. Note that S is a 4×4 matrix. It is desirable to construct a Lorentz invariant quantity out of the components of a bispinor. The previously discussed Lorentz invariant equation, Eqn. 2.6, does not work. However, the following quantity does:

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2 \quad \text{where} \quad \bar{\psi} = \psi^\dagger\gamma^0 \tag{4.7}$$

It is simple to prove that $\bar{\psi}\psi$ is invariant:

$$(\bar{\psi}\psi)' = (\psi')^\dagger\gamma^0\psi' = \psi^\dagger S^\dagger\gamma^0 S\psi = \bar{\psi}\psi \tag{4.8}$$

In this proof the fact that $S^\dagger\gamma^0 S = \gamma^0$ has been utilized; this is easily demonstrated.

2 Plane Wave Solution to the Dirac Equation

Plane wave solutions to the Dirac equation will now be sought. The motivation for this is that these solutions have specific energy and momentum and these observables can be controlled in an experiment. Plane wave solutions have the form:

$$\psi(x) = ae^{-(i/\hbar)x\cdot p}u(p) \tag{4.9}$$

where $u(p)$ is a 4×1 matrix. Substituting $\psi(x)$ into the Dirac equation yields

$$ai\hbar\frac{-i}{\hbar}\gamma^\mu p_\mu e^{-(i/\hbar)x\cdot p}u - mcae^{-(i/\hbar)x\cdot p}u = 0 \tag{4.10}$$

Therefore

$$(\gamma^\mu p_\mu - mc)u = 0 \tag{4.11}$$

Eqn. 4.11 is called the momentum space Dirac equation. This equation has the following solutions [1]:

$$u^{(1)}(E,\mathbf{p}) = N\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)}(E,\mathbf{p}) = N\begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \end{pmatrix}$$

$$v^{(1)}(E,\mathbf{p}) = N\begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} \quad v^{(2)}(E,\mathbf{p}) = N\begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix} \tag{4.12}$$

where N is a normalization constant. By convention, these spinors are normalized as follows:

$$u^\dagger u = \frac{2|E|}{c} \tag{4.13}$$

Using Eqn. 4.13, it can be shown that $N = \sqrt{\frac{|E|+mc^2}{c}}$.

Eqn. 4.12 presents a complete set of solutions to the momentum space Dirac equation: $u^{(1)}$ and $u^{(2)}$ represent two different spin states of a spin-1/2 particle and $v^{(1)}$ and $v^{(2)}$ represent two different spin states of a spin-1/2 antiparticle. It should be noted that $v^{(1)}$ and $v^{(2)}$ are not solutions to the momentum space Dirac equation in the form of Eqn. 4.11. Rather, they are solutions to this equation with the sign of p_μ reversed:

$$(\gamma^\mu p_\mu + mc)v = 0 \tag{4.14}$$

One final fact regarding the spinors u and v, which will be stated but not proven, is that they are complete:

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + mc), \qquad \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - mc) \tag{4.15}$$

3 Feynman Rules for Quantum Electrodynamics

As previously discussed, determining the differential cross section of a particular process requires the amplitude, $\mathcal{M}$, of that process. The amplitude of a process can be calculated through the use of Feynman diagrams. For any given process there are an infinite number of corresponding Feynman diagrams. Each diagram contributes one term to the amplitude of the process. This term is called the amplitude of the diagram and is also denoted $\mathcal{M}$. Each vertex in the diagram contributes a factor of $\sqrt{\alpha}$, the fine structure constant, to the diagram amplitude (cf. Feynman Rule 3, below). The number of vertices in a diagram are said to be the diagram's order. As the fine structure constant is a small number, it is possible to determine the amplitude of a process to an arbitrary precision by keeping only diagrams up to the particular order required to obtain that precision.

The Feynman rules for calculating the amplitude of a Feynman diagram involving quantum electrodynamics will now be presented. The rules will be stated in terms of electrons and positrons but hold for all spin-1/2 fermions. Although not previously discussed, photons will also be included: internal photon lines do not necessitate any further mathematics, however, external photon lines require the use of the photon polarization vector, ϵ. No processes discussed in this text will involve external photon lines and so no discussion of photon polarization will be presented; rules pertaining to external photon lines are included purely for the sake of completeness. For an example of a Feynman diagram, see Fig. 4.1. The Feynman rules for calculating the magnitude of a diagram are [1]:

1. *Notation.* Label the incoming and outgoing four-momenta $p_1, p_2, \ldots, p_n$ and the corresponding spins $s_1, s_2 \ldots, s_n$. Label the internal four-momenta $q_1, q_2, \ldots$. Directions are

assigned to the lines as follows: the arrows on external fermion lines indicate whether it is an electron or a positron - electrons point forward in time, positrons backward; arrows on internal fermion lines are assigned so that every vertex has at least one arrow entering and one arrow leaving. The arrows on external photon lines point forward in time; the choice of direction for internal photon lines is arbitrary.

2. *External Lines.* Incoming electrons contribute a factor of u; outgoing electrons a factor of $\bar{u}$. Incoming positrons contribute $\bar{v}$; outgoing positrons v. Incoming photons contribute ϵ^μ and outgoing photons contribute $\epsilon^{\mu *}$ where, as previously mentioned, ϵ is the photon polarization vector.

3. *Vertex Factor.* Each vertex contributes a factor of

   $$ig_e\gamma^\mu$$

   Where g_e is a dimensionless coupling constant defined as $g_e = -Q\sqrt{4\pi\alpha}$. In this equation, Q is the charge of the particle (rather than the antiparticle) of the particle-antiparticle pair forming the vertex, in units of the positron charge. Thus, for an electron-positron vertex (the third constituent being a photon), $Q = -1$ and so $g_e = \sqrt{4\pi\alpha}$.

4. *Conservation of Energy and Momentum.* Each vertex contributes a delta function of the form

   $$(2\pi)^4\delta(k_1 + k_2 + k_3)$$

   Where the k_i are the four-momenta of the three constituents of the vertex. If a particle is entering the vertex it's k_i is simply its four-momentum; if a particle is leaving the vertex its k_i is the negative of its four-momentum.

5. *Propagators.* Each internal line contributes a factor, called a propagator, as follows

   $$\text{Electrons and Positrons: } \frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$$

   $$\text{Photons: } \frac{ig_{\mu\nu}}{q^2}$$

6. *Integrate Over Internal Momenta.* Each internal momentum, q, contributes a factor:

   $$\frac{d^4q}{(2\pi)^4}$$

   Internal momenta must then be integrated over.

7. *Cancel the Delta Function.* The result will include a factor

   $$(2\pi)^4\delta^4(p_1 + p_2 + p_3 + \ldots - p_n)$$

   corresponding to overall energy-momentum conservation. Cancel this factor.

What remains is a product of several factors. Equate this product to $-i\mathcal{M}$, where $\mathcal{M}$ is the magnitude of the diagram being considered.

Processes can typically be described by many different diagrams. In order to obtain the amplitude for the process the amplitude for each individual diagram must be combined. Typically, the diagram amplitudes are simply added up to obtain the amplitude of the process, however, one additional rule must be added in the case of diagrams which differ only in the exchange of two identical external fermions. This is to account for the antisymmetrization of fermion wave functions:

8. *Antisymmetrization.* A minus sign must be inserted into one of the diagram amplitudes when combining diagrams that differ only in the exchange of two identical incoming or outgoing fermions *or* an incoming electron with an outgoing positron (or vice versa). It does not matter which diagram receives the minus sign as the total amplitude must be squared before it is used in Fermi's Golden Rule.

4 Feynman Rules Example

The Feynman Rules will now be applied to electron-muon scattering. Only diagrams up to second order will be considered. The process is:

$$e^- + \mu^- \rightarrow e^- + \mu^- \tag{4.16}$$

Only one second-order diagram contributes to this process; it is displayed in Fig. 4.1.

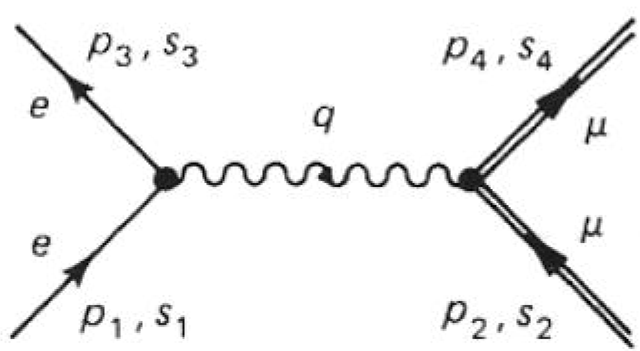

Figure 4.1: Feynman Diagram of Electron-Muon Scattering[1]

Applying the Feynman rules to Fig. 4.1 yields the following factors:

1. The outgoing electron yields a factor $\bar{u}^{(s_3)}(p_3)$
2. The electron-electron vertex yields a factor of $ig_e\gamma^\mu$
3. The incoming electron yields $u^{(s_1)}(p_1)$
4. The internal photon contributes the propagator $\frac{-ig_{\mu\nu}}{q^2}$
5. The outgoing muon contributes $\bar{u}^{(s_4)}(p_4)$
6. The muon-muon vertex yields $ig_e\gamma^\nu$
7. The incoming muon contributes $u^{(s_2)}(p_2)$
8. The electron-electron vertex contributes a delta function $(2\pi)^4\delta^4(p_1 - p_3 - q)$
9. The muon-muon vertex contributes a delta function as well $(2\pi)^4\delta^4(p_2 + q - p_4)$
10. There is only one internal momentum, q, and this contributes $\frac{d^4q}{(2\pi)^4}$. This momenta must be integrated over.

Multiplying all of these factors together yields:

$$(2\pi)^4 \int [\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)u^{(s_1)}(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}^{(s_4)}(p_4)(ig_e\gamma^\nu)u^{(s_2)}(p_2)]\delta^4(p_1-p_3-q)\delta^4(p_2+q-p_4)d^4q$$

Performing the integration, as per Feynman rule 6; dropping the resulting delta function term $(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)$, as per rule 7; and equating to $-i\mathcal{M}$ yields:

$$-i\mathcal{M} = \frac{ig_e^2}{(p_1 - p_3)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)]$$

Thus we can determine the amplitude of the diagram:

$$\mathcal{M} = \frac{-g_e^2}{(p_1 - p_3)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)] \tag{4.17}$$

Squaring:

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1 - p_3)^4}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)][\bar{u}(3)\gamma^\nu u(1)]^*[\bar{u}(4)\gamma_\nu u(2)]^* \tag{4.18}$$

5 Casimir's Trick

As described in Section 3, in order to calculate the amplitude of a Feynman diagram, it is necessary to know the spins and momenta of the incoming and outgoing particles. However, often these quantities are not known. Experiments typically send in particles with random spin and only measure the number of particles scattered in a given direction, not the spin of the scattered particles. Thus, initial spin configurations, i, must be averaged over and the final spin configurations, f, must be summed over:

$$\langle|\mathcal{M}|^2\rangle = \text{average over initial spins, sum over final spins, of } |\mathcal{M}(i \to f)|^2 \tag{4.19}$$

$\langle|\mathcal{M}|^2\rangle$ could then be used in place of $|\mathcal{M}|^2$ in the calculation of the magnitude of a process. One method of determining $\langle|\mathcal{M}|^2\rangle$ is calculating every possible $|\mathcal{M}(i \to f)|^2$ and averaging over initial spin states and summing over final spin states. However, there is a technique which allows this to be done much more efficiently; this technique, called Casimir's Trick, is described below.

As can be seen in the amplitude for electron-muon scattering, Eqn. 4.18, amplitudes contain terms of the form

$$G = [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^* \tag{4.20}$$

where Γ_1 and Γ_2 are 4×4 matrices. It is possible to utilize this form to simplify the calculation of $\langle|\mathcal{M}|^2\rangle$. First, the complex conjugate is evaluated:

$$\begin{aligned}
[\bar{u}(a)\Gamma_2 u(b)]^* &= [u(a)^\dagger\gamma^0\Gamma_2 u(b)]^\dagger \text{ since LHS is a } 1 \times 1 \text{ matrix} \\
&= u(b)^\dagger\Gamma_2^\dagger\gamma^{0\dagger}u(a) \\
&= u(b)^\dagger\gamma^0\gamma^0\Gamma_2^\dagger\gamma^0 u(a) \quad \text{since } \gamma^{0\dagger} = \gamma^0 \text{ and } (\gamma^0)^2 = 1 \\
&= \bar{u}(b)\bar{\Gamma}_2 u(a) \quad \text{where } \bar{\Gamma}_2 = \gamma^0\Gamma_2^\dagger\gamma^0
\end{aligned}$$

Therefore

$$G = [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(b)\bar{\Gamma}_2 u(a)] \tag{4.21}$$

Summing over the spin orientations of particle b

$$
\begin{aligned}
\sum_{b \text{ spins}} G &= \bar{u}(a)\Gamma_1 \left[\sum_{s_b=1,2} u^{(s_b)}(p_b)\bar{u}^{(s_b)}(p_b) \right] \bar{\Gamma}_2 u(a) \quad \text{but using Eqn. 4.15} \\
&= \bar{u}(a)\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2 u(a) \qquad \text{where } \not{p}_b = \gamma^\mu p_\mu \\
&= \bar{u}(a) Q u(a) \qquad \text{where } Q = \Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2
\end{aligned}
$$

Summing over the spin orientations for particle a

$$
\begin{aligned}
\sum_{a \text{ spins}} \sum_{b \text{ spins}} G &= \sum_{s_a=1,2} \bar{u}^{(s_a)}(p_a) Q u^{(s_a)}(p_a) \\
& \quad \text{Or, writing the matrix multiplication explicitly} \\
&= \sum_{s_a=1,2} \bar{u}_i^{(s_a)}(p_a) Q_{ij} u^{(s_a)}(p_a)_j \\
&= Q_{ij} \left[\sum_{s_a=1,2} u^{(s_a)}(p_a)\bar{u}^{(s_a)}(p_a) \right]_{ji} \\
&= Q_{ij}(\not{p}_a + m_a c)_{ji} \\
&= Tr(Q(\not{p}_a + m_a c))
\end{aligned}
$$

Where Tr denotes the trace of the matrix. The definition of the trace and several theorems involving it can be found in the attached appendix. Thus, our final result is:

$$
\sum_{\text{all spins}} [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^* = Tr[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)] \tag{4.22}
$$

Spinors have been eliminated from the calculation. If either μ is replaced by a ν in Eqn. 4.22 then the corresponding mass on the right hand side switches sign.

Casimir's trick will now be applied to the example of electron-muon scattering. Comparing the amplitude of electron-muon scattering, Eqn. 4.18, to the equation for Casimir's trick, Eqn. 4.22, reveals that $\Gamma_1 = \gamma^\mu$ and $\Gamma_2 = \gamma^\nu$. Therefore, $\bar{\Gamma}_2 = \gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu$. The last step utilized the fact that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$, which is easily proven. Thus, applying Casimir's trick twice to Eqn. 4.18 and multiplying by 1/4 to obtain the average:

$$
\langle |\mathcal{M}|^2 \rangle = \frac{g_e^4}{4(p_1 - p_3)^4} Tr[\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu(\not{p}_3 + m_e c)] Tr[\gamma_\mu(\not{p}_2 + m_m c)\gamma_\nu(\not{p}_4 + m_m c)] \tag{4.23}
$$

where m_e is the mass of an electron and m_m is the mass of a muon.

Trace theorems can now be used to simplify Eqn. 4.23:

$$
\begin{aligned}
&Tr[\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu(\not{p}_3 + m_e c)] = \\
&\quad Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) + m_e c[Tr(\gamma^\mu \not{p}_1 \gamma^\nu) + Tr(\gamma^\mu \gamma^\nu \not{p}_3)] + (m_e c)^2 Tr(\gamma^\mu \gamma^\nu)
\end{aligned}
\tag{4.24}
$$

Eqn. 4.24 utilized the fact that the trace is linear, which is stated in Eqn. A-2 and Eqn. A-3 in the appendix. By Eqn. A-5, the terms in square brackets are zero. Eqn. A-7 can be applied directly to the last term to simplify it.

The first term is simplified as follows:

$$
\begin{aligned}
Tr(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) &= (p_1)_\lambda (p_3)_\sigma Tr(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma) \\
&= (p_1)_\lambda (p_3)_\sigma 4(g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu}) \quad \text{using Eqn. A-8} \\
&= 4[p_1^\mu p_3^\nu - g^{\mu\nu}(p_1 \cdot p_3) + p_3^\mu p_1^\nu]
\end{aligned}
\tag{4.25}
$$

Thus:

$$
Tr[\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu(\not{p}_3 + m_e c)] = 4[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m_e^2 c^2 - p_1 \cdot p_3)]
\tag{4.26}
$$

The second trace in Eqn. 4.23 can be determined in the exact same manner, with $m_e \to m_m$, $1 \to 2$, $3 \to 4$, and the Greek indices lowered. Thus

$$
Tr[\gamma_\mu(\not{p}_2 + m_m c)\gamma_\nu(\not{p}_4 + m_m c)] = 4[p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + g_{\mu\nu}(m_m^2 c^2 - p_2 \cdot p_4)]
\tag{4.27}
$$

Substituting Eqn. 4.26 and Eqn. 4.27 into Eqn. 4.23 yields

$$
\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{4g_e^4}{(p_1 - p_3)^4}[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m_e^2 c^2 - p_1 \cdot p_3)] \\
&\quad \times [p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + g_{\mu\nu}(m_m^2 c^2 - p_2 \cdot p_4)] \\
&= \frac{8g_e^4}{(p_1 - p_3)^4}[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \\
&\quad -(p_1 \cdot p_3)(m_m^2 c^2) - (p_2 \cdot p_4)(m_e c)^2 + 2(m_e m_m c^2)^2]
\end{aligned}
\tag{4.28}
$$

6 Cross-section of Electron-Muon Scattering

Using Fermi's Golden Rule and the techniques for determining amplitudes discussed in the preceding sections, it is now possible to determine the cross-section for an actual process. The example that will be considered is electron-muon scattering. The calculation will be performed in the lab frame, assuming the muon is at rest. Also, $m_m \gg m_e$ and so the relevant cross-section formula is Eqn. 3.19:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_m c}\right)^2 \langle|\mathcal{M}|^2\rangle$$

As discussed in Section 3.2, in such a situation the recoil of the lighter particle is negligible. Thus, the four-momenta are:

$$p_1 = (E/c, \mathbf{p}_1) \quad p_2 = (m_m c, 0) \quad p_3 = (E/c, \mathbf{p}_3) \quad p_4 = (m_m c, 0) \tag{4.29}$$

where labels 1 and 3 denote the incoming and outgoing electron, respectively, and 2 and 4 denote the incoming and outgoing muon. Recall that $|\mathbf{p}_1| \simeq |\mathbf{p}_3| = |\mathbf{p}|$. Using the above facts, it is simple to demonstrate the following equations:

$$\begin{aligned}
(p_1 - p_3)^2 &= -4\mathbf{p}^2 \sin^2(\theta/2) \\
(p_1 \cdot p_3) &= m_e^2 c^2 + 2\mathbf{p}^2 \sin^2(\theta/2) \\
(p_1 \cdot p_2)(p_3 \cdot p_4) = (p_1 \cdot p_4)(p_2 \cdot p_3) &= (m_m E)^2 \\
(p_2 \cdot p_4) &= (m_m c)^2
\end{aligned}$$

where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$. Substituting these equations into Eqn. 4.28 yields

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{g_e^2 m_m c}{\mathbf{p}^2 \sin^2(\theta/2)}\right)^2 [(m_e c)^2 + \mathbf{p}^2 \cos^2(\theta/2)] \tag{4.30}$$

Substituting Eqn. 4.30 into the equation for the cross-section, Eqn. 3.19:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{2\mathbf{p}^2 \sin^2(\theta/2)}\right)^2 [(m_e c)^2 + \mathbf{p}^2 \cos^2(\theta/2)] \tag{4.31}$$

This equation is called the Mott formula. It can be used to approximate the differential cross-section for electron-proton scattering in some energy regions. If the incident electron is non-relativistic, $\mathbf{p}^2 \ll (m_e c)^2$, then the Mott formula can be shown to reduce to the Rutherford formula:

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{2mv^2 \sin^2(\theta/2)} \right)^2 \tag{4.32}$$

Chapter 5

Electron-Proton Scattering

1 Elastic Electron-Proton Scattering

If the proton were a point-like particle then electron-proton scattering would be the same as electron-muon scattering but with the muon charge and mass replaced by the mass and charge of the proton. By analogy with electron-muon scattering, the spin-averaged amplitude of the process would be:

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{q^4} L^{\mu\nu}_{\text{electron}} L_{\mu\nu\ \text{proton}} \tag{5.1}$$

where $q = p_1 - p_3$ is the momentum of the virtual photon and

$$L^{\mu\nu}_{\text{electron}} = 2[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m_e^2c^2 - p_1 \cdot p_3)] \tag{5.2}$$

$L_{\mu\nu\text{proton}}$ would be the same but with $m_e \to m_p$, $1 \to 2$, $3 \to 4$, and Greek indices lowered.

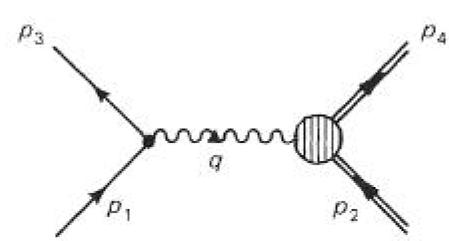

Figure 5.1: Elastic Electron-Proton Scattering [1]

However, the proton is not a point particle and how the proton interacts with the virtual photon is unknown, as is indicated in Fig. 5.1. Thus, $L_{\mu\nu\,\text{proton}}$ must be replaced with the unknown $K_{\mu\nu\,\text{proton}}$.

There are some restrictions on $K_{\mu\nu\,\text{proton}}$, however. It must be a second-rank tensor and can only depend on the variables $p_2 \equiv p$, p_4, and q as these are the only variables present in the problem. Furthermore, $q = p_4 - p$ and so the three variables are not independent - one of them can be eliminated; conventionally, it is p_4. A second-rank tensor can depend on two variables in only the following ways:

$$K^{\mu\nu}_{\text{proton}} = -K_1 g^{\mu\nu} + \frac{K_2}{(m_p c)^2} p^\mu p^\nu + \frac{K_3}{(m_p c)^2} q^\mu q^\nu + \frac{K_4}{(m_p c)^2}(p^\mu q^\nu + p^\nu q^\mu) \tag{5.3}$$

Where the K_i are unknown functions. Factors of $1/(m_p c)^2$ have been introduced to ensure that the K_i all have the same dimensions. It could also be possible to add a term proportional to the antisymmetric combination $(p^\mu q^\nu - p^\nu q^\mu)$ but $L^{\mu\nu}_{\text{electron}}$ is symmetric and so such a term would not contribute to $\langle |\mathcal{M}|^2 \rangle$. The only scalars available for the K_i to depend on are q^2, p^2, and $q \cdot p$. However, $p^2 = m_p c^2$ is a constant and, using conservation of four-momentum, it is simple to demonstrate the following equation:

$$q \cdot p = -q^2/2 \tag{5.4}$$

Therefore, the K_i must be functions of q^2 only. The K_i are also limited by the following equation, derived from conservation of charge [2]:

$$q_\mu K^{\mu\nu} = 0 \tag{5.5}$$

Contracting Eqn. 5.5 separately with q_ν and p_ν yields, after much algebra, two equations:

$$K_3 = \frac{(m_p c)^2}{q^2} K_1 + \frac{1}{4} K_2, \qquad K_4 = \frac{1}{2} K_2 \tag{5.6}$$

Thus, $K^{\mu\nu}$ depends on only two unknown functions of q^2, K_1 and K_2:

$$K^{\mu\nu}_{\text{proton}} = K_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{K_2}{(m_p c)^2} \left(p^\mu + \frac{1}{2} q^\mu \right) \left(p^\nu + \frac{1}{2} q^\nu \right) \tag{5.7}$$

K_1 and K_2 are called form factors. It is possible to determine them experimentally as they are related to the elastic electron-proton scattering cross section. This will now be shown. Substituting Eqn. 5.7 into Eqn. 5.1 in place of $L_{\mu\nu\,\text{proton}}$ yields, after much algebra:

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{2g_e^2}{q^2}\right)^2 \left\{K_1[(p_1\cdot p_3) - 2(m_e c)^2] + K_2\left[\frac{(p_1\cdot p)(p_3\cdot p)}{(m_p c)^2} + \frac{q^2}{4}\right]\right\} \tag{5.8}$$

Using Eqn.5.8 and the appropriate differential cross-section equation, it is possible to determine the differential cross section for elastic electron-proton scattering. The calculation will be performed in the lab frame assuming the target proton is at rest. The collision will be assumed to be moderately energetic, $E_1 \gg m_e c^2$, so the mass of the electron can be neglected, $m_1 \simeq 0$. It should be noted that this assumption is distinct from the assumption used in Mott's equation, which was $E_1 \ll m_{\text{target}} c^2$. Thus, the relevant four-momenta are:

$$p_1 = E_1/c(1, \hat{\mathbf{p}}_1) \quad p_2 = p = (m_p c, 0) \quad p_3 = E_3/c(1, \hat{\mathbf{p}}_3) \tag{5.9}$$

The amplitude can be simplified as follows:

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4 c^2}{4E_1 E_3 \sin^4(\theta/2)}[2K_1 \sin^2(\theta/2) + K_2 \cos^2(\theta/2)] \tag{5.10}$$

where θ is the scattering angle, the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$. The appropriate cross-section equation is that for a massless incident particle in the lab frame, Eqn. 3.20. Substituting the amplitude equation, Eqn. 5.10, into this cross-section equation yields:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{4m_p E_1 \sin^2(\theta/2)}\right)^2 \frac{E_3}{E_1}[2K_1 \sin^2(\theta/2) + K_2 \cos^2(\theta/2)] \tag{5.11}$$

It should be noted that E_3 is not an independent variable, it is determined by conservation of four-momenta. Eqn. 5.11 is known as the Rosenbluth formula. Thus, by measuring the number of particles scattered into a particular solid angle and determining the differential cross section, it is possible to determine the form factors K_1 and K_2. The form factors will not be discussed further as they have been sufficiently developed for the purposes of this text, however, it should noted that the form factors are a popular subject of investigation in themselves as they yield information about the charge distribution within the proton.

It is possible to find form factors such that $K^{\mu\nu}$ reduces to the equation for a muon, Eqn. 4.27. It is easy to demonstrate that the following K_1 and K_2 are sufficient:

$$K_1 = -q^2, \qquad K_2 = 4(m_p c)^2 \tag{5.12}$$

These form factors would accurately describe the proton if it was a point particle, however, it is not and so such an approximation would be a poor one. However, Eqn. 5.12 will prove useful in the discussion of deep inelastic scattering.

2 Inelastic Electron-Proton Scattering

At sufficiently high energies scattering an electron off of a proton may yield a variety of hadrons; such a scattering process is inelastic:

$$e^- + p^+ \rightarrow e^- + X \tag{5.13}$$

where X denotes the various other hadrons that can result from the collision. A second-order Feynman diagram of the process can be found in Fig. 5.2

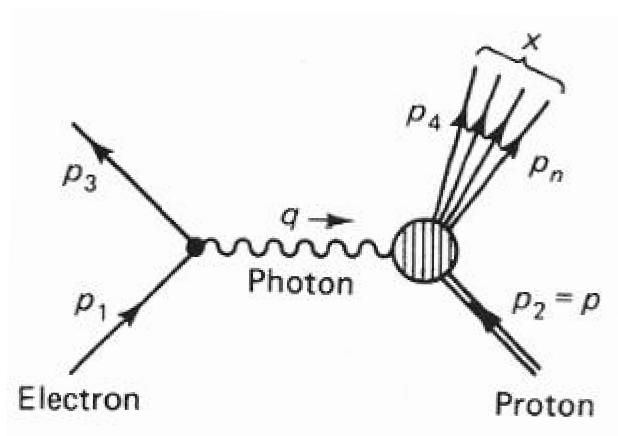

Figure 5.2: Inelestic Electron-Proton Scattering [1]

Much like with the elastic scattering case, the spin-averaged amplitude of the process for a given final state, X, is:

$$\langle |\mathcal{M}|^2 \rangle = \frac{g_e^4}{q^4} L^{\mu\nu}_{\text{electron}} K_{\mu\nu}(X) \tag{5.14}$$

where $K_{\mu\nu}$ is an unknown quantity describing the interaction between the virtual photon, the proton, the the outgoing hadrons. $K_{\mu\nu}$ must depend on $q = (p_1 - p_3)$, $p_2 \equiv p$, and the momenta of the other outgoing particles. The scattering cross section is determined by Fermi's Golden Rule, Eqn. 3.2, with $S = 1$ and $|\mathcal{M}|^2 \rightarrow \langle |\mathcal{M}|^2 \rangle$:

$$
\begin{aligned}
d\sigma &= \frac{\hbar^2 \langle |\mathcal{M}|^2 \rangle}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \left[\left(\frac{c\, d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) \left(\frac{c\, d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \right) \cdots \left(\frac{c\, d^3\mathbf{p}_n}{(2\pi)^3 2E_n} \right) \right] \\
&\quad \times (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 \ldots - p_n)
\end{aligned} \tag{5.15}
$$

Typically, it is only the scattered electron that is detected. Therefore, all of the possible final hadron configurations must be summed over and the momenta of all other scattered particles integrated over. Such measurements are called inclusive. Thus, the cross section for inclusive scattering becomes:

$$
d\sigma = \frac{\hbar^2 g_e^4 L^{\mu\nu}}{4q^2 \sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \left(\frac{c\, d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) 4\pi m_p W_{\mu\nu} \tag{5.16}
$$

where

$$
\begin{aligned}
W_{\mu\nu} &= \frac{1}{4\pi m_p} \sum_X \int \cdots \int K_{\mu\nu}(X) \left(\frac{c\, d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \right) \cdots \left(\frac{c\, d^3\mathbf{p}_n}{(2\pi)^3 2E_n} \right) \\
&\quad \times (2\pi)^4 (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 \ldots - p_n)
\end{aligned} \tag{5.17}
$$

Once again calculations will be performed in the lab frame with an incoming electron striking a proton at rest. The electron is assumed to be moderately energetic, $E_1 \gg m_e c^2$ therefore it can be assumed that $m_1 = m_e = 0$. Thus, $\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = m_p E_1$. Changing to spherical coordinates, $d^3\mathbf{p}_3 = |\mathbf{p}_3|^2 d|\mathbf{p}_3| d\Omega$, and performing a change of coordinates to energy using $|\mathbf{p}_3| = E_3/c$ reduces Eqn. 5.16 to:

$$
\frac{d\sigma}{dE_3 d\Omega} = \left(\frac{\alpha\hbar}{cq^2} \right)^2 \frac{E_3}{E_1} L^{\mu\nu} W_{\mu\nu} \tag{5.18}
$$

Unlike in the elastic scattering case, E_3 is not determined by conservation of momentum as the extra hadrons resulting from the collision can carry away various amounts of momentum. Thus, what must be considered is the differential equation in a particular energy range dE_3, as shown in Eqn. 5.18.

As before, $W_{\mu\nu}$ must be a second rank tensor constructed out of q and p as follows:

$$
W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{(m_p c)^2} p^\mu p^\nu + \frac{W_3}{(m_p c)^2} q^\mu q^\nu + \frac{W_4}{(m_p c)^2} (p^\mu q^\nu + p^\nu q^\mu) \tag{5.19}
$$

where the W_i must be functions of q^2, p^2, or $q \cdot p$. As in elastic scattering, p^2 is a constant and so can be disregarded. In elastic scattering, it was also possible to disregard $q \cdot p$, however, due to the fact the total momentum of X is not constrained as it was before, $p_{\mathrm{X}}^2 = p_4^2 = m_p^2 c^2$, it is not possible to find a relation connecting q^2 and $q \cdot p$. Thus, the W_i must be functions of q^2 and $q \cdot p$.

Due to gauge invariance [2], the W_i have the following constraint:

$$q_\mu W^{\mu\nu} = 0 \tag{5.20}$$

From which the following equations can be derived, with much difficulty:

$$W_3 = \frac{(m_p c)^2}{q^2} W_1 + \left(\frac{q \cdot p}{q^2}\right)^2 W_2, \qquad W_4 = -\frac{(q \cdot p)}{q^2} W_2 \tag{5.21}$$

Therefore, Eqn. 5.19 reduces to:

$$W^{\mu\nu} = W_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2}{(m_p c)^2} \left[p^\mu - \left(\frac{q \cdot p}{q^2}\right)^2 q^\mu\right] \left[p^\nu + \left(\frac{q \cdot p}{q^2}\right)^2 q^\nu\right] \tag{5.22}$$

$W_1(q^2, q \cdot p)$ and $W_2(q^2, q \cdot p)$ are called structure functions. Substituting Eqn. 5.22 into Eqn. 5.18 and performing much simplification yields:

$$\frac{d\sigma}{dE_3 d\Omega} = \left(\frac{\alpha \hbar}{2E_1 \sin^2(\theta/2)}\right)^2 \left[2W_1 \sin^2(\theta/2) + W_2 \cos^2(\theta/2)\right] \tag{5.23}$$

Eqn. 5.23 is the fundamental result for inelastic electron-proton scattering. Using this equation, it is possible to determine W_1 and W_2 from scattering experiments.

As was previously mentioned, W_1 and W_2 are functions of q^2 and $q \cdot p$, however, it is convenient to define a new variable, called Bjorken x and defined $x = -q^2/(q \cdot p)$. Thus, W_1 and W_2 can be made to be functions of q^2 and x rather than q^2 and $q \cdot p$.

Elastic scattering can be considered a special case of inelastic scattering, where $p_{\mathrm{total}}^2 = m_p^2 c^2$. Thus, it should be possible to find a W_1 and W_2 such that Eqn. 5.23 reduces to the Rosenbluth formula, Eqn. 5.11. It can be shown that the following W_1 and W_2 are sufficient:

$$W_{1,2}(q^2, x) = -\frac{K_{1,2}(q^2)}{2m_p q^2} \delta(x - 1) \tag{5.24}$$

where E_3 must be integrated over to obtain the Rosenbluth formula. It should be noted that the delta function in Eqn. 5.24 demands that $x = 1$. This is logical as $x = -q^2/(q \cdot p) = 1$ implies that $q \cdot p = -q^2/2$, which is the relationship between q^2 and $q \cdot p$ for elastic scattering, Eqn. 5.4.

It was previously noted that the K_i took on a particular form if the proton was assumed to be a point particle, expressed in Eqn. 5.12. Substituting these form factors into Eqn. 5.24 yields the equation for the W_i if the scattering is assumed to be elastic and the target is a point particle of mass m:

$$W_1 = \frac{1}{2m}\delta(x-1), \qquad W_2 = -\frac{2mc^2}{q^2}\delta(x-1) \tag{5.25}$$

Chapter 6

Deep Inelastic Scattering

Having derived equations for the differential cross section of electron-proton scattering, it is now possible to develop a physical model for the W_i and compare it to experimental data.

1 The Quark Hypothesis

It is now proposed that the proton is composed of pointlike, spin-1/2 particles - quarks; this is the quark hypothesis. If this hypothesis is true, at sufficiently high energies it should be possible to model inelastic electron scattering off a proton as the sum of several elastic scatterings with free quarks. Electron-proton scattering at these energies is called deep inelastic scattering. In this section, the implications of this hypothesis for the proton structure functions will be investigated.

As was discussed previously, when the target is assumed to be a point particle and the scattering is elastic, the structure functions take on the form expressed in Eqn. 5.25. Substituting in the mass and charge of a quark of flavour i, these equations become:

$$W_1^i = \frac{Q_i^2}{2m_i}\delta(x_i - 1), \qquad W_2^i = -\frac{2m_i c^2 Q_i^2}{q^2}\delta(x_i - 1) \tag{6.1}$$

Where m_i is the mass of the quark, p_i is the quark's momentum, and $x_i = -q^2/(q \cdot p)$. Here, Q_i is the charge of the quark written in units of the positron charge; thus, Q_i for an up quark is 2/3 and for a down quark is -1/3. Q_i would typically be included in the vertex factor, as per Feynman Rule 3, however in this case it is included in the formulae for the W_i; this is to leave the equation for the cross section, Eqn. 5.23, unchanged.

Suppose that z_i is the fraction of the proton four-momentum carried by the target quark:

$$p_i = z_i p \tag{6.2}$$

Therefore, since $p_i^2 = m_i^2 c^2$ and $p^2 = m_p^2 c^2$:

$$m_i = z_i m_p \tag{6.3}$$

This implies that $x_i = x/z_i$. Thus,

$$W_1^i = \frac{Q_i^2}{2m_p}\delta(x - z_i), \qquad W_2^i = -\frac{2m_p c^2 Q_i^2}{q^2}\delta(x - z_i) \tag{6.4}$$

Let $f_i(z_i)$ be the probability that the ith quark carries fraction z_i of the incident proton momentum,p. To obtain the structure functions of the proton, all of the quark flavours must be summed over and the z_i integrated over. Thus:

$$W_1 = \sum_i \int_0^1 \frac{Q_i^2}{2m_p}\delta(x - z_i) f_i(z_i) = \frac{1}{2m_p}\sum_i Q_i^2 f_i(x) \tag{6.5}$$

$$W_2 = \sum_i \int_0^1 \frac{-2x^2 m_p c^2}{q^2} Q_i^2 \delta(x - z_i) f_i(z_i) = \frac{-2m_p c^2}{q^2} x^2 \sum_i Q_i^2 f_i(x) \tag{6.6}$$

Therefore, it is possible to define:

$$F_1(x) \equiv m_p W_1 = \frac{1}{2}\sum_i Q_i^2 f_i(x) \tag{6.7}$$

$$F_2(x) \equiv -\frac{q^2}{2m_p c^2 x} W_2 = x \sum_i Q_i^2 f_i(x) \tag{6.8}$$

Therefore, in energy regions where the quark hypothesis holds the structure functions $W_1(q^2, x)$ and $W_2(q^2, x)$ reduce to the the new functions, $F_1(x)$ and $F_2(x)$. Restated, the quark hypothesis implies that there is an energy region where the functions F_1 and F_2, as defined in Eqn. 6.7 and Eqn. 6.8, become independent of q^2. This phenomena is called Bjorken scaling.

Furthermore, comparing Eqn. 6.7 and Eqn. 6.8 reveals a relation between the two new functions $F_1(x)$ and $F_2(x)$:

$$F_2(x) = 2xF_1(x) \tag{6.9}$$

Eqn. 6.9 is called the Callan-Gross relation. This relation implies that the point-particle that the electron is colliding with within the proton must have spin 1/2. If the target particle had spin-0 rather than spin-1/2 then the relation would be $\frac{2xF_1}{F_2} = 0$ [1]. Substituting Eqn. 6.7, Eqn. 6.8, and Eqn. 6.9 back into the relevant differential cross section equation, Eqn. 5.23, yields

$$\frac{d\sigma}{dE_3 d\Omega} = \frac{F_1(x)}{2m_p}\left(\frac{\alpha\hbar}{E_1 \sin(\theta/2)}\right)^2 \left[1 + \frac{2E_1E_3}{(E_1 - E_3)^2}\cos^2(\theta/2)\right] \tag{6.10}$$

Thus, the differential cross section is now dependent on only one unknown function, $F_1(x)$. As shown in Eqn. 6.7, determining F_1 requires that the $f_i(x)$ be known. Determining the $f_i(x)$ will be the subject of Section 3 of this chapter.

The most obvious benefit of Bjorken scaling and the Callan-Gross relation is that they have simplified the equation for the differential cross section, making it depend only one unknown function of one variable. However, they also provide an experimental test by which the quark hypothesis can be tested. Whether or not these predicted phenomena correspond with observation will be the subject of the subsequent section.

2 Experimental Evidence for Bjorken Scaling and the Callan-Gross Relation

Bjorken scaling advances that there must exist some range of energies for which the functions F_1 and F_2 are independent of $Q^2 = -q^2$. Displayed in Figure 6.1 is an experimental plot of F_2 versus Q^2 for various values of x. As can be seen from this plot, at $x = 0.14$ the plot is approximately a flat line, demonstrating that F_2 is independent of Q^2 in this region. This is evidence of Bjorken scaling. It must also be noted that, at higher values of x, the Q^2 dependence of F_2 returns. This phenomenon, known as Bjorken scaling violation, is due to the presence of gluons in the proton. Gluons will be discussed further in the next section.

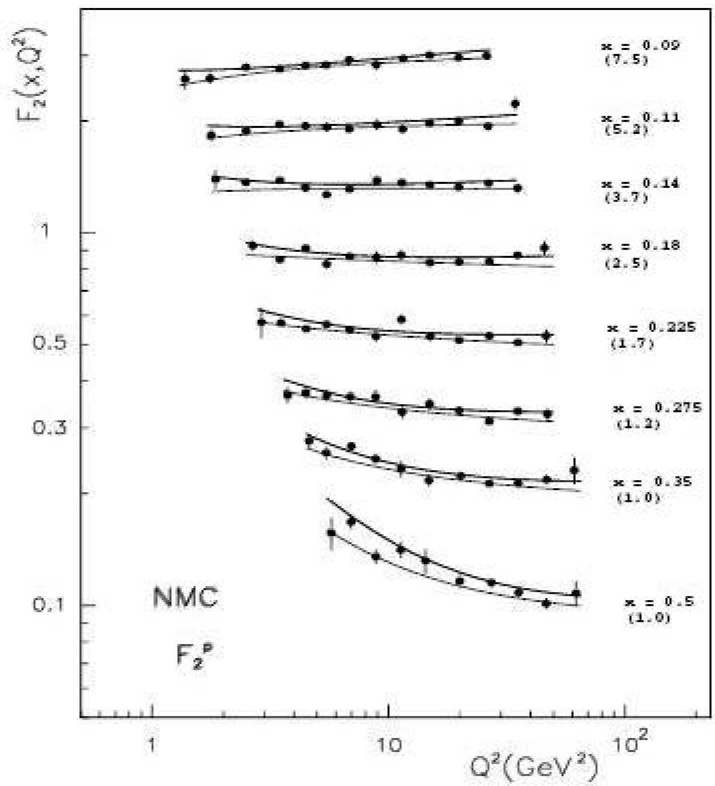

Figure 6.1: A plot of F_2. The data in each x-series has been scaled by the number in parentheses to separate the values. Error bars represent statistical uncertainties, solid lines systematic uncertainties. [3]

Displayed in Fig. 6.2 is an experimental plot of $\frac{2xF_1}{F_2}$ versus x. If the Callan-Gross relation, Eqn. 6.2, holds then the plot should be a constant with a value of 1.

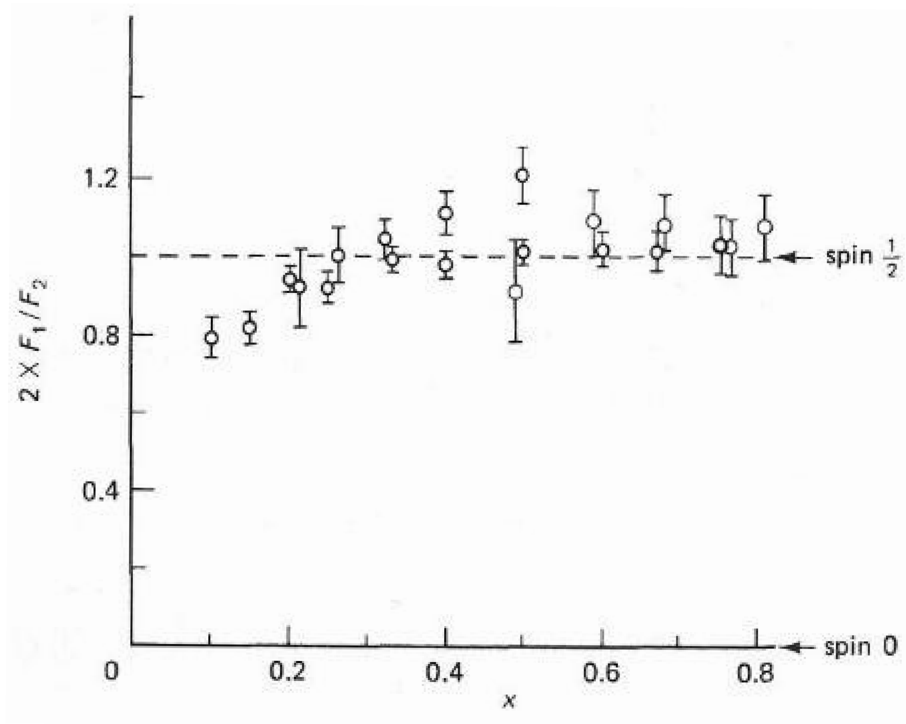

Figure 6.2: A plot of $2xF_1/F_2$. The Callan-Gross relation holds well for $x \geq 0.2$. Plot taken from [1] with data from [4].

The plot in Fig. 6.2 is approximately constant, thus providing evidence for the Callan-Gross relation. Therefore, both Bjorken scaling and the Callan-Gross relation are supported by the experimental evidence. As both are consequences of the quark hypothesis, it would seem that the experimental evidence supports such a model of the proton.

3 Gluons and Sea Quarks

Eqn. 6.3 implies that the momentum fraction carried by a quark is determined by its mass. Thus, the $f_i(z_i)$ must take the form of a delta function:

$$f_i(z_i) = \delta(m_i/m_p - z_i) \tag{6.11}$$

Utilizing Eqn. 6.11 and supposing that the proton consists of two up quarks and a down quark, as is conventionally done, Eqn. 6.7 and Eqn. 6.8 become:

$$F_1(x) = \frac{1}{2}\left[\left(\frac{2}{3}\right)^2 \delta\left(\frac{m_u}{m_p} - x\right) + \left(\frac{2}{3}\right)^2 \delta\left(\frac{m_u}{m_p} - x\right) + \left(\frac{-1}{3}\right)^2 \delta\left(\frac{m_d}{m_p} - x\right)\right]$$

$$F_2(x) = x\left[\left(\frac{2}{3}\right)^2 \delta\left(\frac{m_u}{m_p} - x\right) + \left(\frac{2}{3}\right)^2 \delta\left(\frac{m_u}{m_p} - x\right) + \left(\frac{-1}{3}\right)^2 \delta\left(\frac{m_d}{m_p} - x\right)\right] \quad (6.12)$$

Where m_u and m_d are the mass of the up quark and the down quark, respectively. Displayed in Fig. 6.3 are schematics of F_1 and F_2 versus x. Clearly, these plots do not agree with Eqn. 6.12. Thus, our hypothesis that the photon has interacted with free quarks must be flawed.

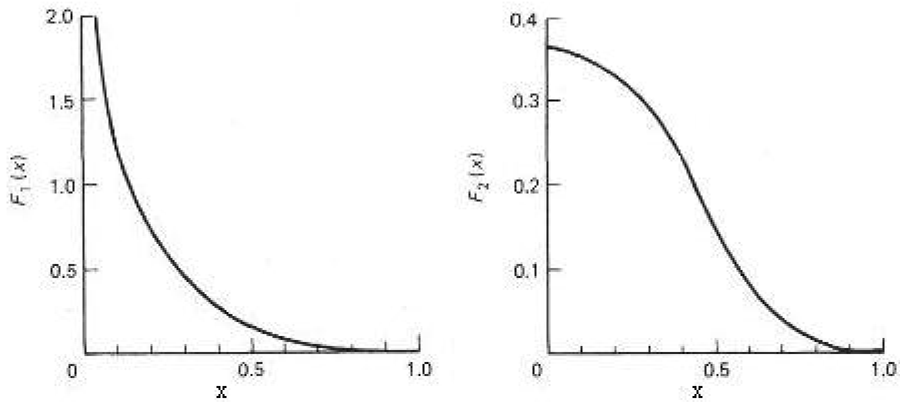

Figure 6.3: A plot of $F_1(x)$ and $F_2(x)$ [1]

Upon further reflection, it is a poor approximation to consider the quarks free. As the quarks are confined within the proton, they must be continuously interacting through the strong force. Furthermore, Eqn. 6.11 assumes that the mass of a quark is well defined. However, the quarks are bound within the nucleus and so their mass would not be specified as the mass of a free quarks would.

It will now be demonstrated that our hypothesis is also flawed in claiming that the proton is entirely made up of quarks. Let $u(x)$ be the probability that momentum fraction x is carried by an up quark and $d(x)$ represent the same quantity for down quarks. Thus,

$$F_1(x) = \frac{1}{2}\left[\left(\frac{2}{3}\right)^2 u(x) + \left(\frac{-1}{3}\right)^2 d(x)\right] \quad (6.13)$$

Using our previous model, Eqn. 6.11, $u(x)$ and $d(x)$ would have taken the form $u(x) = 2\delta(m_u/m_p - x)$ and $d(x) = \delta(m_d/m_p - x)$. It may seem plausible to suppose that $u(x) =$

$2d(x)$, as there are twice as many up quarks and the two flavours are approximately the same mass. However, this is not supported by experimental data, as shown in the plots in Fig. 6.4. In both plots, $u(x) = 2d(x)$ seems to hold near $x = 0.1$, however, it clearly fails near $x = 1$ in Fig 6.4 (b) and near $x = 0$ in from Fig. 6.4 (a).

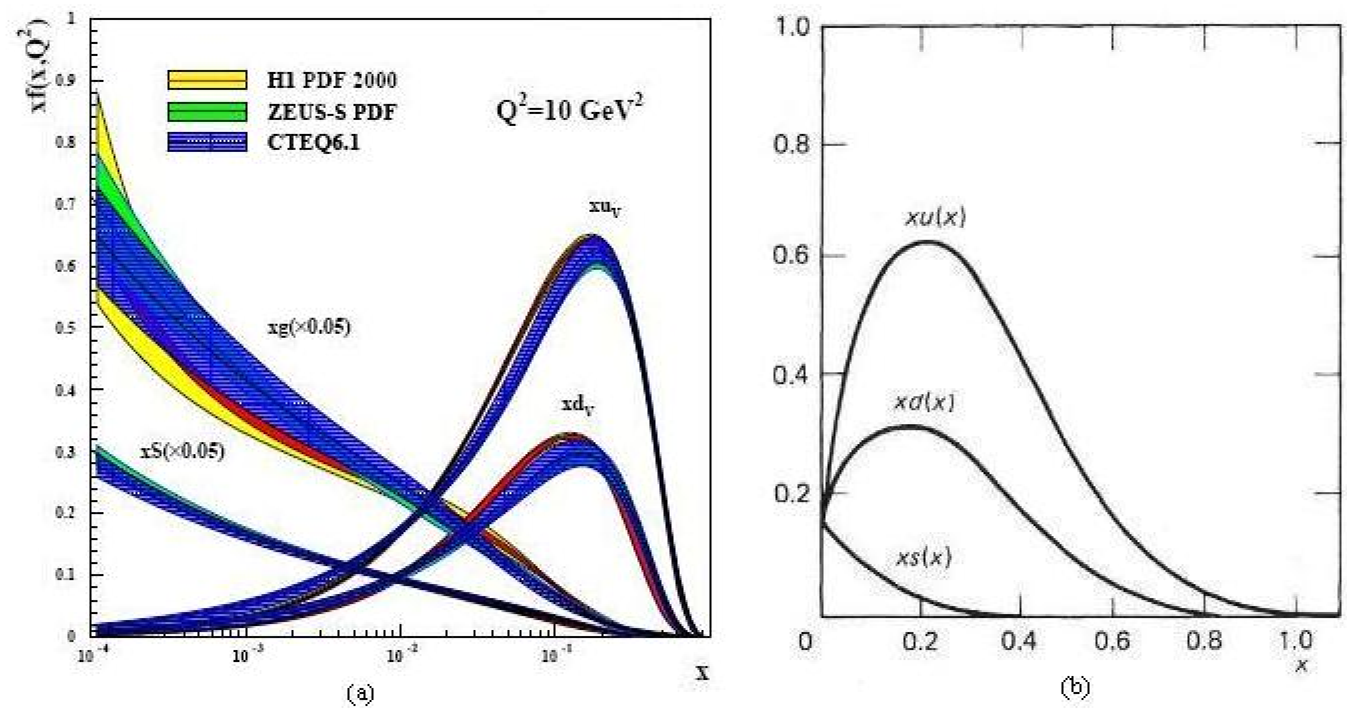

Figure 6.4: Two plots depicting $u(x)$ and $d(x)$. (a) is taken from [5] and (b) from [1]

Alternatively, it could be proposed that the average momenta carried by up quarks is twice that carried by down quarks. Thus:

$$\int_0^1 xu(x)dx = 2\int_0^1 xd(x)dx \tag{6.14}$$

Eqn. 6.14 is supported by data from electron-neutron scattering [1]. Thus

$$
\begin{aligned}
\int_0^1 F_2(x)dx &= \int_0^1 \left[\frac{4}{9}xu(x) + \frac{1}{9}xd(x)\right] dx \quad \text{from Eqn. 6.13} \\
&= \frac{4}{9}\int_0^1 xu(x)dx + \frac{1}{9}\int_0^1 xd(x)dx \\
&= \frac{8}{9}\int_0^1 xu(x)dx + \frac{1}{9}\int_0^1 xd(x)dx \quad \text{using Eqn. 6.14} \\
&= \int_0^1 xd(x)dx
\end{aligned}
$$

Integrating under the area of the experimental curve in Fig. 6.3 yields $\int_0^1 F_2(x)dx = 0.18$ [1]. Thus

$$
\int_0^1 xd(x)dx = 0.18, \qquad \int_0^1 xu(x)dx = 0.36 \tag{6.15}
$$

Therefore the average total momentum carried by the up and down quarks is:

$$
\int_0^1 pxu(x)dx + \int_0^1 pxd(x)dx = 0.18p + 0.36p = 0.54p \tag{6.16}
$$

Thus, only 54% of the momentum of the incoming proton is accounted for by the proposed quarks. Consequently, there must be other constituent particles of the proton that are carrying the missing momentum. As was previously mentioned, the quarks interact through the strong force and so they must be exchanging gluons. These gluons are capable of carrying momentum. Furthermore, the gluons are capable of forming quark-antiquark pairs. The original three quarks, the two up and a down, are called valence quarks, and these gluon-formed quarks are called sea quarks. Thus, Eqn. 6.16 provides evidence for the presence of gluons within the proton.

4 Conclusion

A theory of scattering has been presented and used to describe electron-proton scattering. It has been shown how the quark hypothesis has the physical implications of Bjorken scaling and the Callan-Gross relation. Experimental evidence was presented supporting these claims. Furthermore, the fact that the three quarks postulated to form the proton do not account for all of the momentum in the proton has provided evidence for the presence of gluons in the proton. These were the goals put forth in the introduction and as they have been completed, this text now draws to a close.

Appendix: Trace Theorems

The trace of a square matrix A, denoted $Tr(A)$, is defined as follows:

$$Tr(A) = \sum_i A_{ii} \tag{A-1}$$

Presented here are list of theorems involving the trace that will be used in the text of this thesis[1]:

$$\begin{aligned}
Tr(A+B) &= Tr(A) + Tr(B) && \text{(A-2)}\\
Tr(\alpha A) &= \alpha Tr(A) && \text{(A-3)}\\
Tr(AB) &= Tr(BA) && \text{(A-4)}\\
& \text{The trace of the product of an odd number of} \\
& \qquad \text{gamma matrices is 0} && \text{(A-5)}\\
Tr(1) &= 4 && \text{(A-6)}\\
Tr(\gamma^\mu \gamma^\nu) &= 4g^{\mu\nu} && \text{(A-7)}\\
Tr(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) &= 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda}) && \text{(A-8)}
\end{aligned}$$

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